MATHEMATICAL MODELING OF GRADED RIVER PROFILES[1]

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ABSTRACT

Numerical modeling of the longitudinal profiles of rivers at grade is accomplished using the basic equations of open-channel flow, sediment transport equations, and empirical relations for downstream variation in flow discharge, sediment discharge, sediment caliber, and channel width. Only in some cases are the computed stream profiles fit exactly by any one of the commonly supported mathematical function analogs to graded profile form—exponential, logarithmic, or power function, but in most cases any of these functions can provide a fit with a degree of error smaller than would be noted in treating field data. Profiles dominated by spatial change in fluid and sediment discharge are distinctly power functions, while profiles dominated by sediment size reduction are not necessarily exponential in form. Other important controls on profile shape are the degree of downstream width change in response to increasing discharge and the general range of sediment size. A dynamic model of a river's approach to grade indicates that disequilibrium river profiles closely approximate a graded profile shape even while the general slope is relatively high, and significant erosion remains to achieve equilibrium.

INTRODUCTION

The longitudinal profiles of many graded rivers have smooth, concave-upward curves similar to graphic curves produced by a variety of simple mathematical functions. Several of these functions have been adopted by different researchers as the actual descriptors of the profile form of the graded river, but none of these equations has been shown to be generally applicable in the field. The strongest arguments for particular mathematical analogs to graded profiles (cited later) derive those functions by rational analysis of controls on the river system in equilibrium, with the mathematics of the analysis determining the curve form that stream profiles should at least approximate.

Some rivers that are seemingly graded have longitudinal profiles which deviate strongly from a smooth curve, likely due to local influence of major tributaries, discontinuous changes in sediment caliber downstream, pool and riffle sequences, effects of geology on vegetation and groundwater regime, and so on. It is tempting to think that such "noise" may not only be responsible for obvious deflections in particular river profiles, but also for the general lack of correspondence between fairly smooth river profile curves and mathematical functions proposed to describe those profiles. But the question remains open whether even an ideally graded river profile will match the curve form of any fairly simple mathematical function.

There is ample reason to pursue the rational description and prediction of river profiles. If we can adequately describe the ideal, graded forms for particular rivers, then deviations from these forms can be considered as significant sources of information. Local deviations can be investigated as signs of tectonic uplift or tilting, as in work by Volkov et al. (1967). Rational equations for river profiles are used in paleogeomorphological reconstruction from stream terraces or fluvial sediments, with the profile being extrapolated upstream to predict slopes and paleorelief in source areas, or downstream to determine base level. A number of mathematical models that have been developed to describe progressive changes in stream channels and their sediments assume *a priori* that particular, simple functions adequately describe slope change downstream. For example, both the model reported by Strahler (1952, p. 936–937) for long-term river channel lowering and that developed by Hamblin et al. (1981) for short-term channel erosion due to uplift are based on assumptions that river profiles are exponential curves.

The purpose of this study is to explore a fluvial mathematical model of river profiles

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that is significantly more comprehensive than previously employed, in order to address the following objectives: (1) Calculate ideal, graded river profiles as functions of water discharge, sediment caliber, and sediment discharge, and test the approximation of these profiles by the mathematical curves commonly proposed for the prediction of graded river profiles. (2) Calculate the forms of river profiles during their approach to grade in order to understand properly what erosional and form characteristics rivers display in that state.

In the present analysis the definition of grade is one where the river has achieved a mass equilibrium and is in a state of no additional morphometric change. The river is viewed in "graded time" (Schumm and Lichty 1965), where inflows of water and sediment are considered as constant through time, barring the effects of major climatic changes. The effects of individual flood events are not of interest, and a single discharge value is taken to represent the hydrologic regime for each point along the river.

COMMONLY PROPOSED RELATIONS DESCRIBING PROFILE FORM

The graded stream is one which has become adjusted in form to transport both water and sediment discharges supplied by its drainage over a period of years (Mackin 1948). Therefore a minimal set of controlling variables for the form of a graded stream includes discharge, sediment load, and sediment characteristics such as mean grain size. Past attempts to produce mathematical expressions for equilibrium stream profiles may be classified into three groups by their approach to this set of variables.

The first group neglects two of the variables in favor of a single controlling, or at least dominant, variable. Those accepting grain size variation as the controlling factor rationally derive exponential profile forms (Sternberg 1875; Shulits 1941; Yatsu 1955). Those taking discharge as the controlling factor develop empirically calibrated power functions for slope which give profiles that are either logarithmic or are themselves simple power functions (Gilbert 1877; Leopold and Maddock 1953; Carlston 1968).

Rather than ignore important controlling variables, the second group includes both discharge and grain size, and sometimes sediment load, combined in empirical equations (Lane 1937; Hack 1957). Additional controlling variables have been proposed to supplement or replace the above, including ones controlling channel geometry (Rubey 1933) and pattern (Schumm 1960). This general approach yields equations for channel slope that are combinations of power functions.

The third group avoids primary consideration of any of the controlling variables, proceeding by analogy with physical systems other than the fluvial system. These include diffusivity-type equations (Scheidegger 1970, p. 209–210) giving exponential profiles, random walks (Leopold and Langbein 1962) yielding power-function and logarithmic profiles, and the balance of conditions of minimum work and equal work distribution (Langbein and Leopold 1964) producing profiles bounded by power functions. Yang (1971*a*) uses potential energy considerations to derive graded profiles that deviate from all three of the curve forms given above. These heuristic approaches have the advantage of appealing to general principles governing the stream system, principles that the whole group of stream variables considered individually by other workers must conform to. Nonetheless, they stand or fall upon the researcher's wisdom to choose an analogy that is correct and complete.

In short, research to derive mathematical functions for graded river profiles has failed to treat explicitly even the most minimal set of controlling variables other than by entirely empirical means. In various studies, however, mathematical curves of exponential, logarithmic, and power function forms has each been fit to actual river profiles with success (Woodford 1951, Hack 1957, Butakov 1970, Shepherd 1985). Such variation in field results ought to be explained by a comprehensive analysis of stream longitudinal profile form.

A STEADY-STATE MODEL

A mathematical model describing unsteady, nonuniform flow through an erodable channel can be written in terms of four dependent variables (fig. 1), the cross-sectionally averaged flow velocity, u; water surface elevation above a datum, h; mean bed surface elevation above a datum, b; and the

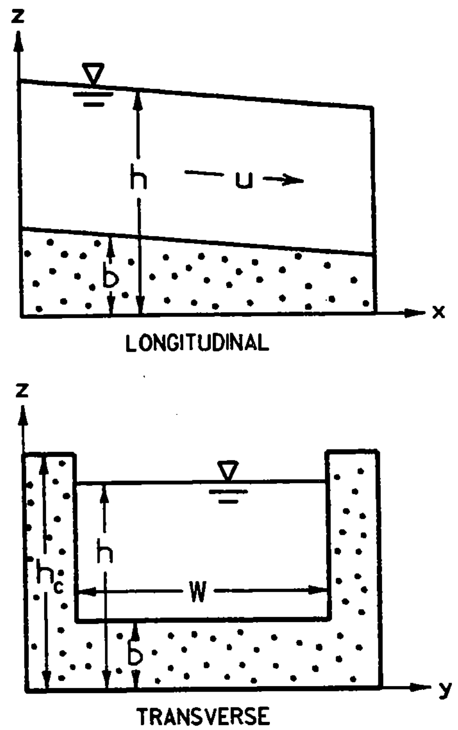

FIG. 1.—Definitions of dependent variables in the model. Both bed surface elevation b and flow velocity u are cross-sectionally averaged. The constant h_c, representing bank elevation, is factored out in production of the steady-state model.

channel width, W. Each of these variables is a function of two independent variables: distance downstream, x, and time, t. Equations comprising the model are the following:

– conservation of momentum for clear water

$$\frac{\partial[uW(h-b)]}{\partial t} + \frac{\partial[u^2W(h-b)]}{\partial x} + gW(h-b)\frac{\partial h}{\partial x} = -Wgu|u|n^2(h-b)^{-1/3} \quad (1)$$

– continuity of water

$$\frac{\partial[W(h-b)]}{\partial t} + \frac{\partial[uW(h-b)]}{\partial x} = Q_i \quad (2)$$

– continuity of sediment

$$-\alpha\frac{[W(h_c-b)]}{\partial t} + \frac{\partial Q_{st}}{\partial x} = Q_{si} \quad (3)$$

where g is the acceleration of gravity, n is Manning's roughness coefficient, Q_i is lateral inflow of water per unit channel length, Q_{si} is lateral inflow of sediment in volume per unit length, α is one minus the porosity of channel sediment, h_c is a constant bank elevation, and Q_{st} is bed material transport in volume per unit time, a function of hydraulic conditions and sediment characteristics.

Equation (1) is employed with the assumption that suspended sediment loads will not significantly alter the effective fluid density. Other approximations are ones common in the literature of open-channel flow for natural channels with simple cross sections: assumption of hydrostatic pressure in the fluid, neglect of the momentum coefficient, and consideration of a hydraulically wide channel with width-depth ratio greater than 10. Metric units are assumed in writing the flow resistance term found on the right.

If no information about stream modification through time is desired, but only the graded profile and associated flow geometry data, the equation set can be greatly simplified. In graded time, the equilibrium profile is unchanging and the hydraulics are those of steady, nonuniform flow. All time derivatives in the above equations drop out, leaving the set of ordinary differential equations:

$$\frac{d[u^2W(h-b)]}{dx} + gW(h-b)\frac{dh}{dx} = -Wgu|u|n^2(h-b)^{-1/3} \quad (4)$$

$$\frac{d[uW(h-b)]}{dx} = Q_i \quad (5)$$

$$\frac{dQ_{st}}{dx} = Q_{si} \quad (6)$$

With downstream flow taken to be in the positive x direction, equations (4) and (5) combine to yield:

$$\frac{1}{2}\frac{du^2}{dx} + g\frac{dh}{dx} = -gu^2n^2(h-b)^{-4/3} - \frac{u}{W(h-b)}Q_i \quad (7)$$

The last term on the right of equation (7) represents the inertial effects of lateral inflows, assumed to be entering the channel at right angles, reducing stream velocity. With lesser angles of inflow this effect would be reduced. In the applications at hand the relative numerical value of this term is negligible, and it is dropped from consideration. The total energy slope is defined (Henderson 1966):

$$S_e = -\frac{d}{dx}\left(h + \frac{u^2}{2g}\right) \qquad (8)$$

and is substituted into (7), yielding:

$$S_e = u^2 n^2 (h - b)^{-4/3} \qquad (9)$$

which is Manning's Equation for gradually varied flows:

$$u = \frac{1}{n}(h - b)^{2/3} S_e^{1/2} \qquad (10)$$

If an external accounting is invoked to sum the effects of sediment inflows, equation (6) integrates to:

$$Q_{st} = Q_s \qquad (11)$$

where Q_s is the sum of inflows increasing along the channel length and Q_{st} is replaced by an appropriate sediment transport relation. If discharge is also externally integrated, equation (5) gives the discharge formula:

$$Q = Wu(h - b) \qquad (12)$$

The familiar equations (10), (11), and (12) relate energy slope and the four dependent variables to externally specified, controlling values of discharge, sediment discharge, and, via the transport relation, sediment characteristics. Because channel width is to be taken as a function of discharge (Leopold and Maddock 1953), width is here treated as independent. Although relationships between hydraulic roughness and other individual stream variables are well known, no generally recognized synthesis of these exists. Leopold and Maddock (1953) provide evidence that Manning's n typically shows little systematic variation downstream, and it will be taken as an externally imposed constant in this study.

Substituting D (mean depth) for $(h - b)$, we have a set of three equations with three unknowns. Hydraulic variables written into various sediment transport equations are u, D, shear stress τ, dimensionless shear stress θ, shear velocity U_*, and stream power Ω. With equations (10) and (12), all of these may be written in terms of energy slope and externally specified variables:

$$u = n^{-0.6}\left(\frac{Q}{W}\right)^{0.4} S_e^{0.3} \qquad (13)$$

$$D = \left(\frac{Qn}{W}\right)^{0.6} S_e^{-0.3} \qquad (14)$$

$$\tau = \rho g D S_e = \rho g\left(\frac{Qn}{W}\right)^{0.6} S_e^{0.7} \qquad (15)$$

$$\theta = \frac{\tau}{(s - 1)\rho g d} = \frac{1}{(s - 1)d}\left(\frac{Qn}{W}\right)^{0.6} S_e^{0.7} \qquad (16)$$

$$U_* = \left(\frac{\tau}{\rho}\right)^{1/2} = g^{0.5}\left(\frac{Qn}{W}\right)^{0.3} S_e^{0.35} \qquad (17)$$

$$\Omega = \rho g Q s_e \qquad (18)$$

where ρ is fluid density, s is specific gravity of sediment, and d is sediment diameter. These relations may be substituted into the sediment transport equation of choice to obtain an equation relating the energy slope to specified sediment discharge and other independent variables. For most modern transport relations this requires an iterative solution, with slope values being obtained for discrete points along the stream as in the similarly-conceived step techniques for backwater calculation in hydraulic engineering.

The algorithm for calculation of a graded profile begins with a sweep from upstream to downstream points, with specified variables and energy slope being calculated for each successive data point. Solution for energy slope is by Newton iteration, using the slope at the immediately upstream point as an initial guess. A water surface level is specified at the downstream end, and with mean velocity calculated by equation (13), the downstream elevation of the total energy line, H, is calculated.

$$H = h + \frac{u^2}{2g} \qquad (19)$$

Averaged energy slope is then projected upstream from point to point with calculated values of velocity and depth allowing solution for water and bed surface elevations. This algorithm has been coded into the FORTRAN program GRADE.

The model equations do not account for differential transport of a distribution of particle sizes at any one point along the modeled stream and therefore do not allow for selective longitudinal sorting of bed material. Rather, a downstream change in mean or representative grain size is taken as an independently imposed condition.

GRADE allows calculation of ideal, graded, profile curves conditioned by a variety of controlling variables and particular sediment transport relations. The model is consistent with general physical laws of conservation of mass and momentum. While not all degrees of freedom of the river system are allowed to come into play, it improves on previous rational models in that downstream variations in discharge, sediment caliber, sediment discharge, and channel form are included as factors controlling the form of the graded profile.

CONTROLLING DATA VALUES

A numerical solution method such as that employed in this model cannot provide a general equation describing graded river profiles. In response to specific values of the controlling variables, it can provide specific values of stream gradient that can be assembled into a profile. The exploration of the model must proceed by a series of "numerical experiments." In this initial study it is important to choose specific controlling variable values, and ranges of values, which will be near those actually found in natural streams and be appropriate to the assumptions involved in the model. Later analysis can show how sensitive the general results are to the specific values chosen.

Mathematical expressions are necessary as well to describe downstream changes in values of the externally specified, controlling variables such as discharge and sediment caliber. The set of such expressions assembled in this section represents a consensus of the literature; where a range of opinions exist regarding a particular variable's downstream change, that range is represented by inclusion of more than one mathematical expression.

Basin Area and Channel Length.—Hack (1957) observes that for a number of drainage basins in the northeastern United States the relationship between mainstream length (L) and drainage area (A) can be expressed by the proportionality:

$$L \propto A^{0.6} \tag{20}$$

A similar relation between stream lengths and areas can be derived by example calculations from Horton's (1945) laws of stream lengths and drainage areas. Using typical values for the length ratio and area ratio (Scheidegger 1970, p. 147), a set of lengths and areas for streams of different orders can be obtained, and plotted on double-log paper. The resulting curve rapidly converges to a line with slope of about 0.6. Subsequent work (Gray 1961; Leopold et al. 1964; Mueller 1973), including data from a variety of climates, supports Hack's conclusion for basin areas up to about 10,000 km^2 but indicates that for greater basin areas the exponent gradually is reduced to 0.5. Shreve (1974) applies topological methods to produce a curve analog to these results, but the curve is not expressed by a simple mathematical function.

It seems most appropriate for this study to treat streams with drainage areas for which the length-area relation is best specified, those for which equation (20) applies. A stream 100 km in length, which by Hack's original data corresponds to a basin area of about 1500 km^2, is well within that range. Because in the present analysis the upstream end is defined as a point where water and sediment discharges are zero, and because it is inappropriate to expect first-order stream segments to abide by many of the generalizations applied here, only the 90 km segment extending from 10 km to 100 km downstream will be mathematically modeled and analyzed.

Discharge.—Empirical relations between discharge and drainage area are widely reported as simple power functions. The present analysis does not treat arid region streams, where discharge may decrease in the downstream direction and where a conceptual link between discharge and drainage area is suspect. Hack (1957) compares average discharge to drainage area for various points in the Potomac River basin, deriving the simple relation:

$$Q \propto A^{1.0} \tag{21}$$

However, higher discharges occurring less frequently typically do not increase downstream in direct proportion to increasing drainage area. For example, Aron and Miller (1978) report the following average relation between annual maximum flood peak and drainage area for approximately 50 streams in Pennsylvania and New Jersey:

$$Q \propto A^{0.7} \tag{22}$$

Strahler (1964, p. 50) indicates that for various locations and discharge frequencies of interest, the exponent generally falls in the range 0.5 to 1.0.

We will generally take the view here supported at length by Wolman and Miller (1960) and widely held today (Richards 1982), that the "channel-forming discharge" is one well above average flow but still exceeded a number of times in a decade, roughly corresponding to bankfull flow. However, with geographic variation in the area-discharge exponent for flows of that range, it is reasonable to adopt two different values, those shown in equations (21) and (22), bracketing the range and allowing for variation of empirical results.

In order to express discharge as a function of stream length, equation (20) is inverted, replacing L with x,

$$A \propto x^{1.7} \tag{23}$$

and combined with (21) and (22) to give respectively, in non-dimensionalized form,

$$\frac{Q}{Q_{max}} = \left(\frac{x}{x_{max}}\right)^{1.7} \tag{24}$$

$$\frac{Q}{Q_{max}} = \left(\frac{x}{x_{max}}\right)^{1.2} \tag{25}$$

where x_{max} is the full length of the stream, 100 km here, and Q_{max} is the discharge at the downstream end.

The specific value of discharge used here must be chosen from a great range of possibilities. A runoff rate of 20 cm per unit area per year, typifying less humid areas of the eastern U.S., would yield a mean annual discharge of about 10 m^3/s from a basin of 1500 km^2. The ratio of discharge near bankfull to mean annual discharge is not constant from site to site, but calculations from data given by Leopold et al. (1964) and Carlston (1965) for several stream locations in the east-central U.S. (table 1) indicate that the ratio of 10 is well within the range. So the outflow from the modeled basin is set at 100 m^3/s. A semiarid region stream would have considerably lower mean annual discharge but also might be expected to have less frequent, more extreme floods dominating channel form. Though this argument, and its inverse for more humid climates, is only qualitative, it does suggest the model results may be applicable to streams outside the narrow

TABLE 1

RATIOS OF FLOWS NEAR BANKFULL TO MEAN ANNUAL DISCHARGE FOR RIVER LOCATIONS IN THE EAST-CENTRAL U.S.

River Location	Drainage Area (km^2)	Discharge Ratio
Wabash R., New Corydon, Ind.	670	6
Eel R., North Manchester, Ind.	1080	6
Wildcat Creek, Owasco, Ind.	1010	9
Fall Creek, Millersville, Ind.	810	13
Driftwood Ck., Edinburg, Ind.	2730	9
Bogue Chitto, Tylertown, Miss.	1300	8
Allegheny R., Eldred, Pa.	1420	6
Licking R., Farmers, Ky.	2150	9
Red R., Clay City, Ky.	940	16
LaCrosse R., West Salem, Wis.	1030	7
Trempeleau R., Dodge, Wis.	1660	8
Buffalo R., Tell, Wis.	1050	10
Kickapoo R., LaFarge, Wis.	690	17
Pecatonica R., Darlington, Wis.	710	24
Grant R., Burton, Wis.	690	27

SOURCES.—Data from Leopold et al. 1964, table 7–13, Carlston 1965, table 1.

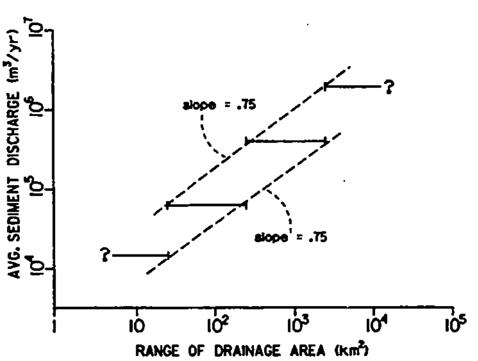

FIG. 2.—Graphic representation of data summary given by Gottschalk (1964) for sediment delivery by drainage basins in four size ranges. Original sediment discharge data, given per unit area, has been multiplied by a "median" drainage area for each range, taken to be a value within the range that is one-half log cycle from the range end points. The trend of these values suggests that sediment discharge is approximately proportional to the three-fourths power of drainage area.

climatic range implied by the numbers used above.

Sediment Discharge.—The relation of sediment discharge to drainage area is also typically expressed as a simple power function,

$$Q_s \propto A^m \tag{26}$$

with various researchers giving values for the exponent m that vary from 0.6 to 1.1 (Gottschalk 1964). Gottschalk's own compilation of average sediment production rates for order-of-magnitude ranges of basin area, involving 1100 basins in the U.S., plots on double-logarithmic paper to give an exponent of approximately 0.75 (fig. 2). Schumm (1963) supports an exponent value of 0.85 for the relation, based mainly on data from the western U.S. Again it is reasonable to represent the range of opinion by employing two values for the exponent, 1.0 and 0.7. Combined with the length-area relation, these yield equations, parallel in form to those for discharge,

$$\frac{Q_s}{Q_{sm}} = \left(\frac{x}{x_{max}}\right)^{1.7} \tag{27}$$

$$\frac{Q_s}{Q_{sm}} = \left(\frac{x}{x_{max}}\right)^{1.2} \tag{28}$$

with Q_{sm} representing the sediment discharge at the downstream end.

Data such as Gottschalk's, with sediment production expressed per unit area, are commonly employed to support a belief that the ratio of sediment discharge to water discharge decreases in the downstream direction. Comparison of equations (24) and (25) for water discharge to those above for sediment indicates such a belief is not necessarily true. The pairs of equations for water and sediment discharge were developed independently, and this study will include numerical experiments based on all four possible combinations: one in which the ratio decreases significantly downstream, two in which the ratio remains constant, and one in which the relative sediment load significantly increases downstream. Although the last of these is difficult to judge for a whole river length, the results may be instructive for interpretation of individual river segments.

Data relating sediment discharge during bankfull flow to mean sediment discharge are not widely available. Nixon (1959) estimates from British rivers that bankfull discharge is equalled or exceeded 0.6% of the time, and Wolman and Miller (1960, p. 59) give plots of cumulative percentage of total sediment load versus percentage of time corresponding flows are equalled or exceeded for four river locations in the western U.S. From these we can derive percentage of total sediment load carried by flows equalled or exceeded 1% of the time but less than flows exceeded 0.3% of the time, giving examples of the desired ratio (table 2). Interpolating a similar plot for Brandywine Creek in Delaware from two data points that Wolman and Miller supply, we derive a ratio similar to the others. Based on these limited results, for this work sediment discharge is set at 20 times the mean sediment discharge. Assuming production of 100% solid material equivalent to the fairly low denudation rate of 5 cm/1000 yrs (Bloom 1978, p. 284–286), a 1500 km^2 basin yields a mean sediment outflow of approximately 0.0023 m^3/s. So "bankfull" sediment discharge at the downstream end is set at 0.045 m^3/s.

Sediment Transport Functions.—It is our intention to insert into the model sediment transport equations which have strong rational backing, consistent with the present-day geomorphological emphasis on control of sediment movement by stream power (Yang 1971*b*; Bull 1979), but which also have been

TABLE 2

RATIOS OF PERCENTAGE OF SEDIMENT TRANSPORT ACCOMPLISHED TO PERCENTAGE OF TIME FOR FLOWS NEAR THE DISCHARGE EQUALLED OR EXCEEDED 0.6% OF THE TIME, FIVE U.S. STREAMS

River Location	% of total load carried within 0.7% of the time	Ratio: % of load to % time
Rio Puerco at Rio Puerco, N.M.	29	41
Cheyenne R. near Hot Springs, S.D.	26	37
Colorado R. at Grand Canyon, Ariz.	9	13
Niobrara R. near Cody, Nebr.	5	7
Brandywine Ck. at Wilmington, Del.[a]	13	19

SOURCE.—Wolman and Miller 1960, fig. 2.
[a] Relation interpolated from two data points.

calibrated with empirical data. Particular equations are appropriate only to particular ranges of sediment size. Two equations, one appropriate to sand sizes and one to fine gravel, serve in separate model runs in this analysis.

Yang (1973, eqn. 26) presents a relation for total load, based on the concept of unit stream power, verified and calibrated using a massive amount of flume data and lesser amount of field data, with sediment sizes ranging from 0.2 to 2 mm. The accuracy of this equation compared to other commonly applied relations has been verified for a variety of independent data sets (Yang and Molinas 1982; Task Committee 1982; Borah el al. 1982). The threshold for sediment transport is expressed by a critical velocity, U_{cr} (Yang 1973, eqns. 18 and 19). The sediment transport calculation also requires estimation of particle settling velocities from sieve diameters, provided in this study by a corrected empirical equation (Baba and Komar 1981, eqns. 2 and 5).

In the remaining numerical experiments involving fine gravel sizes, the Engelund bedload equation (Engelund and Fredsøe 1976, eqn. 13) is applied, using the Shields curve (Simons and Senturk 1976, p. 410) to estimate θ_c, the critical dimensionless shear stress for sediment motion. Multiple shear stress terms combine in the equation to make transport proportional to stream power, and the equation has been calibrated with flume data, particle sizes ranging from 1 to 8 mm.

Sediment Caliber.—There are two general approaches to be taken concerning change of sediment sizes in the downstream direction. Data sets such as those given by Hack (1957) and Brush (1961) include examples of drainage basins with trends of sediment size remaining constant, increasing, and decreasing downstream over tens of kilometers, typically with a large amount of scatter in the data points. Although much of the variability can be explained in terms of geologic controls, one is nevertheless left feeling that, when attempting to model generalized graded river profiles, one should impose no downstream change in caliber at all, because there is no clear, consistent trend to follow.

However, many workers (Sternberg 1875; Shulits 1941; Pettijohn 1957; Scheidegger 1970; Tanner 1971) support a systematic caliber change downstream of the form

$$\frac{d}{d_o} = e^{-ax} \qquad (29)$$

where d_o is sediment diameter at a specified upstream point ($x = 0$) and a is a number of units inverse to the units of x (e.g., km^{-1}), considered as a diminution coefficient, where the diminution may be due to wear and/or sorting. Here we take the "exponential law of gradation" (Scheidegger 1970) as an empirical observation with no implication of specific cause.

Representative values for the coefficient a, collected from a variety of sources, appear in table 3. Yatsu (1955) reports a discontinuity in the relation for Japanese rivers as multigrain igneous pebbles break down to individual sand grains, with average coefficient values for sands below the discontinuity being lower than for the gravels. It seems best to reflect this by adopting two different representative coefficients: for numerical experi-

TABLE 3

GRAIN SIZE REDUCTION COEFFICIENTS FOR VARIOUS STREAMS

Locality	Grain Size Range (mm)	Coefficient a, (km^{-1})
Rhine River[a]	150–100	.0032
7 Japanese rivers	100–20	.043[b]
River Mur	70–35	.0064
3 English rivers	70–15	.053[b]
River Fowley, U.K.	65–35	.028
Rapid Creek, S.D.	24–6	.029
5 Japanese rivers	2–.4	.023[b]
Mississippi River	.6–.2	.0008

SOURCES.—Yatsu 1955; Pettijohn 1957; Knighton 1980; Richards 1982.
[a] Largest cobbles only; all other values derived for mean grain sizes.
[b] Mean value of several coefficients.

ments involving downstream change in fine gravel sizes a is given the value 0.02, and for experiments involving downstream reduction of sands a is set at 0.003.

The specific ranges of grain size chosen for the numerical experiments should be close to the size ranges for which the equations were calibrated. A d_o value of 15 mm, chosen for the gravel model runs, yields sediment sizes reducing to 2 mm at the downstream end. A d_o of 0.4 mm for the sand-range runs yields sizes reducing to 0.3 mm over the 100 km. Runs with no spatial variation in caliber for gravel and sand ranges are given constant sediment sizes of 6 mm and 0.4 mm.

Channel Width.—Spatial change in channel width with spatial increase in discharge is given by hydraulic geometry relations (Leopold and Maddock 1953) and can be expressed in the following form:

$$\frac{W}{W_{max}} = \left(\frac{Q}{Q_{max}}\right)^Z \tag{30}$$

where W_{max} represents width at the downstream end. Leopold and Maddock give the exponent Z an average value of 0.5 for mean annual discharges. Exponents derived for humid-region streams at bankfull discharge are somewhat below this value (Wolman 1955; Carlston 1965). Width-discharge relations for mean annual discharges and two-year floods in the Missouri River basin (Osterkamp and Hedman 1982) indicate exponent values closer to 0.75 for sand and gravel-bed streams, with little systematic variation in the values due to sediment size. The value 0.5 is sufficient as a general exponent for this study.

With discharge at the downstream end 100 m^3/s, channel width at that point is set at 50 m, a value biased toward humid-region, single-channel streams. For the numerical experiments described in the following section, this gives a width-depth ratio of about 35 at the most downstream piont.

Miscellaneous Constants.—As indicated in the section on model development, Manning's roughness coefficient has been shown empirically to change little on average in the downstream direction. This study incorporates a constant value of 0.03, representative of a fairly clear, natural channel.

Specific gravity of sediment is taken to be 2.65, equal to that of quartz. Calculations pertaining to both Yang and Engelund transport functions require values of fluid viscosity, which are primarily affected by temperature. Taking temperature as 15°C, kinematic viscosity is 1.15×10^{-6} m^2/s, and dynamic viscosity is 0.01139 poise. The choice of values for each of these variables must be regarded as potentially influencing the form of derived, graded profiles. This possibility will be addressed by sensitivity analysis in a later section.

GENERAL MODEL RUNS

In collecting a set of controlling variable values to allow calculation of idealized, graded river profiles, we have adopted two ways for discharge to vary downstream, two ways to vary downstream sediment discharge, two ranges of sediment size, and two general approaches to spatial changes in sediment size. Sixteen numerical experiments using GRADE have been run, treating all possible combinations of these alternatives. Longitudinal profiles, controlling variable functions, and identification numbers for these runs are shown schematically in figure 3. The profiles are drawn from computed bed surface elevations and are made "dimensionless," utilizing fractions of maximum profile elevations and lengths. Actual mean slope values for these profiles range from 0.001 to 0.002, and the range of Froude numbers for all locations in the runs is 0.32–0.72,

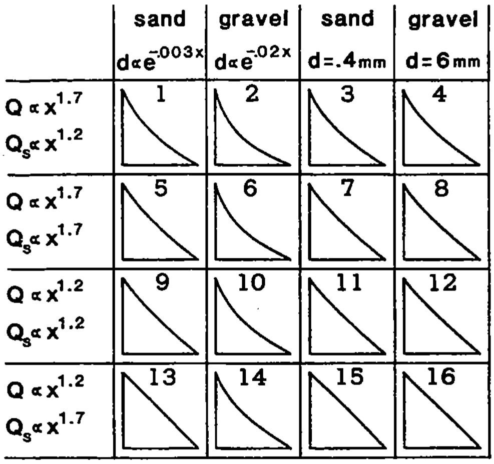

FIG. 3.—Stream bed profile forms calculated in general model runs 1–16, with imposed downstream changes in flow discharge Q, sediment discharge Q_s, and sediment diameter d applicable to each of the runs shown on vertical and horizontal axes of the matrix. The variable x is distance downstream, for change in d specified to be in units of kilometers. Profiles have each been vertically exaggerated to a square format to allow best comparison of shape.

with values for the majority of the profiles not exceeding 0.5.

The results indicate that profile concavity is significantly affected by all three of the major controlling variables. The highest degree of concavity is produced in cases with a strong alongstream decrease in sediment caliber, runs 2, 6, and 10. Concavity is enhanced by a strong change in discharge downstream and weak change in sediment load. In runs 13–16, where sediment load increases downstream relative to discharge, profiles in gravel have reduced concavity and profiles in sand become convex. We have calculated values for profile concavity (table 4) as ratios of areas measured on the profile graphs:

$$C_A = \frac{A_1}{A_2} \tag{31}$$

where A_1 is the numerically integrated area that lies between the profile curve and a straight line connecting the profile endpoints and A_2 is the triangular area below that straight line and above a horizontal axis connecting with the profile's downstream endpoint. If the profile curve is convex, then A_1 is taken as a negative value. This method allows concavity to be derived consistently, regardless of curve form.

Before testing the fit of basic exponential, logarithmic, and power functions to these ideal profiles, one must estimate the degree of error produced in the profile calculations due to the use of numerical approximation techniques. There are two potential sources of such error. The first is the use of a numerical method to match an appropriate energy slope to the imposed sediment discharge. The convergence criterion for this process requires that the sediment load be calculated correctly to more than 10 significant digits, discounting this as a source of any discernible error.

The second potential source of error is in the use of a finite set of points to represent the river profile, with slopes averaged between these points used to project the profile upstream in steps. Re-computing run 1 several times using different numbers of data points (fig. 4) gives evidence that use of 91 data points, 1/km of modeled stream length, is a reasonable balance between accuracy and computation time costs. With a 901-point run used as basis for comparison, the 91-point run generates 2.5 cm of error at the upstream end. The upstream accumulation of this error is not in proportion to the profile's increase in elevation and, therefore, produces a minor change in profile form. The general degree of error along the profile can be expressed by a standard deviation of residuals

$$\sigma = \left[\frac{\Sigma(\Delta h)^2}{N-1}\right]^{0.5} \tag{32}$$

where Δh's are elevation error values along the profile, and N is the number of such values. The standard deviation is conveniently expressed in the same units as the variable being tested for error, and the standard deviation derived for the 91-point run is somewhat less than 1 cm.

MODELED PROFILES COMPARED TO MATHEMATICAL FUNCTIONS

In order to test how well the computed profiles are fit by simple exponential, logarithmic, and power functions, we have

TABLE 4

STANDARD DEVIATIONS FOR REGRESSION CURVES BASED ON THREE SIMPLE FUNCTIONS, PROFILES RESULTING FROM GENERAL MODEL RUNS

Run No.	Sediment Caliber (mm) (x in km)	Profile Concavity C_A	Standard Deviations, σ (m) for fit of simple functions: Exponential	Power	Log.
($Q \propto x^{1.7}$, $Q_s \propto x^{1.2}$)					
1	$d_o = .4, d \propto e^{-.003x}$	.233	1.52	.0334[a]	.493
2	$d_o = 15, d \propto e^{-.02x}$	.340	2.40	.416	.372[a]
3	$d = .4$	.214	1.60	.0002[b]	.681
4	$d = 6.0$	.187	1.28	.0073[b]	.664
($Q \propto x^{1.7}$, $Q_s \propto x^{1.7}$)					
5	$d_o = .4, d \propto e^{-.003x}$	.122	.544	.0275[a]	.340
6	$d_o = 15, d \propto e^{-.02x}$	.292	2.00	.319[a]	.517
7	$d = .4$	.102	.579	.0023[b]	.428
8	$d = 6.0$	.107	.736	.0527[a]	.587
($Q \propto x^{1.2}$, $Q_s \propto x^{1.2}$)					
9	$d_o = .4, d \propto e^{-.003x}$	.091	.344	.0245[a]	.238
10	$d_o = 15, d \propto e^{-.02x}$	.236	1.36	.273[a]	.491
11	$d = .4$	.072	.370	.0012[b]	.302
12	$d = 6.0$	.069	.408	.0220[a]	.352
($Q \propto x^{1.2}$, $Q_s \propto x^{1.7}$)					
13	$d_o = .4, d \propto e^{-.003x}$	−.009	.0803	.0371[a]	.186
14	$d_o = 15, d \propto e^{-.02x}$	.196	1.26	.332[a]	.762
15	$d = .4$	−.029	.0698	.0005[b]	.0791
16	$d = 6.0$	.002	.340	.225[a]	.284

[a] Best fit to profile, of three tested functions.
[b] "Exact" fit to profile, with error less than that potentially imposed by model numerical solution method.

performed regressions using the functions as regression models:

exponential function

$$y = \beta_1 e^{\beta_2 x} + \beta_3 \quad (33)$$

power function

$$y = \beta_1 (x + \beta_2)^{\beta_3} + \beta_4 \quad (34)$$

logarithmic function

$$y = \beta_1 \log(x + \beta_2) + \beta_3 \quad (35)$$

where y is elevation and β's are regression coefficients independently determined for each function and each profile. In some cases previous workers have applied these equations with fewer coefficients, in effect requiring that curve origins coincide with either their measurements' zero-elevation datum, or the horizontal location chosen as the stream's upstream end, or both. Although the nature of the present study allows the stream's upstream end to be specified with some assurance, regressions here are allowed to develop curves of the various forms with any origins that will give best fit to the data.

The best fit is defined as that which minimizes the sum of squares of residuals, which also gives a minimum standard deviation of

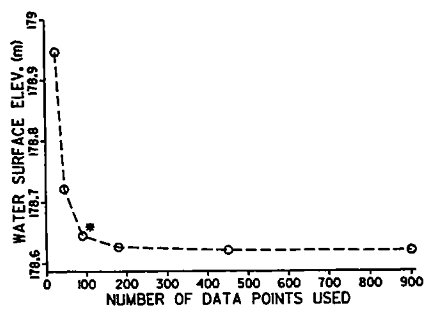

FIG. 4.—Relation of calculated river profile elevation at upstream end (x = 10 km) to number of points of calculation used along the profile for run 1 of the model. The number used in this study, 91 (*, on diagram), gives an elevation value 2.5 cm higher than that derived using 10 times as many data points.

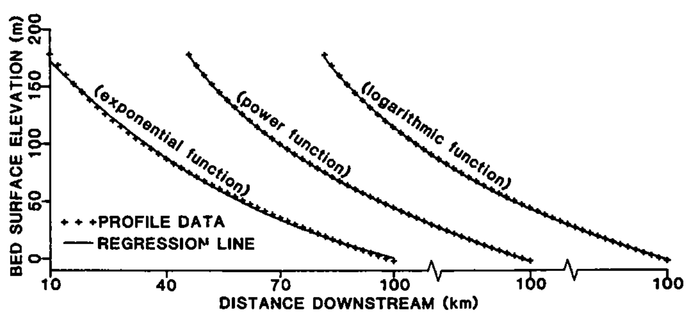

FIG. 5.—Best-fit regression curves for exponential, power function, and logarithmic statistical models applied to profile data generated in run 1. The profile data, included in each case for comparison, show that all three regression models can provide close, though not exact, fits.

residuals (i.e., standard error of estimate). Standard deviation values, expressing lack of fit between simple curve analogs and the 16 computed profiles, are listed in table 4. Where profile curves are convex ($C_A < 0$), the logarithmic function standard deviations shown are the results of a modified regression model, with $-x$ substituted for x in equation (35).

Perhaps the most notable aspect of these results is that the degree of fit is generally high. For profiles ranging approximately 100–200 m in relief, even a systematic error of 1 or 2 m might well be considered acceptable, and consistent deviations measured in centimeters would be entirely masked in field data (fig. 5). If profiles such as these were encountered in the field, any of the three preferred profile equations, considered individually, could be taken to provide a reasonable analog to profile form. Yet when relative values of standard deviation are compared, the power function is almost universally to be preferred. In cases with little or no change in sediment caliber downstream, other than the practically straight profiles of runs 13 and 16, the power function yields errors at least an order of magnitude less than do the other functions. In those cases where sand is a constant size downstream, the error is less than potential errors of the numerical solution method, a condition we will refer to as an exact fit. Where gravels strongly decrease in size downstream, the errors associated with the power function increase, presumably due to the influence of the exponential form of caliber reduction. In these cases it is the logarithmic function that rivals and in one case exceeds the power function in accuracy. The fit provided by the exponential function remains relatively poor. Results are no different when water surface data are substituted for bed surface data in the regressions.

Three caliber-related factors may control how closely particular computed profiles approximate power functions. Change of grain size, noted above, is one factor. There is also a difference associated with general grain size, sands allowing a better fit than gravels. But the difference in grain size is also associated with a difference in the equations used to model sediment transport. Thus the question arises: how much of the difference in modeled profiles results from differences in theories of sediment transport, and how much from some other effect related to grain size but independent of equation used? One approach to this question is to recompute runs 4, 8, and 12 using the Yang equation instead of the Engelund equation. The resulting profiles cannot be considered appropriate to streams in nature, but unrealistic runs such as these may nevertheless aid understanding of the mathematical model and its limitations. Use of the Yang equation to model gravel transport in these runs yields profiles fit by power functions with standard deviations very close to those associated with the Engelund equation. This suggests that it is the difference in sediment size rather than the choice of transport equation that affects the profile approximation to a simple power function.

ADDITIONAL, SIMPLIFIED RUNS

The results given in the previous section have been analyzed primarily by assigning particular variations in profile form to the in-

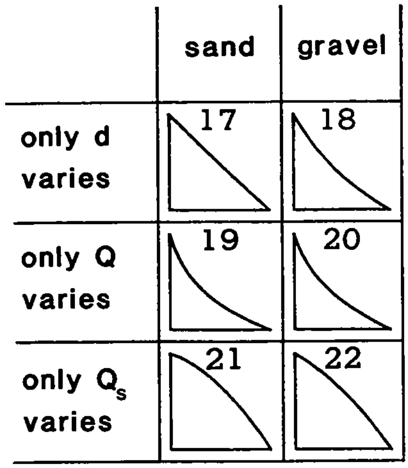

FIG. 6.—Stream bed profile forms calculated in runs 17–22, where in each case only one controlling variable value is allowed to vary normally in the downstream direction. Other variables are maintained at constant, close-to-average values. The downstream variations in flow discharge Q, sediment discharge Q_s, and sediment diameter d used here are the same as those applied simultaneously in general runs 9 and 10.

fluence of particular controlling variables and their variations downstream. Having constructed a rational model allowing a number of variables to influence graded profile form, it is natural to wish to isolate their effects. This is done directly via a series of additional numerical experiments.

Effects of Major Variables.—The influence of an individual variable on profile form is examined by holding all other variables constant. We have run a set of six experiments including cases of isolated downstream change in discharge, sediment discharge, and sediment caliber, treating each case for both sand and gravel sizes. When water discharge or sediment discharge is the variable allowed to change in the downstream direction, its increase is set proportional to $x^{1.2}$; otherwise these controls are given constant values respectively of 30 and 0.0136 m^3/s. The constant and varying values for sediment size are those employed previously. Profiles for these six runs, 17–22, appear in figure 6.

Runs 17 and 18 make apparent the influence on profile form of the two exponents, both within reported ranges, adopted to represent sediment caliber diminution downstream. Regression results (table 5) show a greater difference, with the practically straight profile in sand (run 17) being almost exactly fit by an exponential function, whereas the profile in gravel is not closely fit by any function, though the logarithmic function affords a slightly better fit than the other two. Even sole influence by exponentially decreasing sediment size does not develop a distinctively exponential profile. Runs with discharge and sediment discharge individually controlling form yield profiles that are essentially power functions, following the form of equations describing downstream variation in those variables. The strongly convex profiles produced in runs 21 and 22 supplement the evidence in the general runs 1–16 that downstream variation in sediment discharge relative to flow discharge has potential to influence significantly profile form.

Transport-Threshold Profiles.—In runs accomplished to this point, profiles have been modeled which can support a significant, imposed sediment discharge. Sediment caliber has primarily come into play as a factor influencing the slopes required to provide this capacity. Some workers (Holmes 1952; Yatsu 1955) imply competence to be the more important requirement influencing profile form. We mathematically express competence in the form of critical transport criteria such as U_{cr} and θ_c, and must set sediment discharge close to zero to allow the control of sediment size on these factors to become a significant control on profile form.

Ten additional numerical experiments (fig. 7) duplicate the conditions of all previously described ones, with the exception that Q_s, set proportional to Q, has a value at the downstream end of 0.0004 m^3/s. This gives a sediment load about 1% of that in previous runs. The numerical solution method does not allow us to set Q_s at zero. Regression results on these profiles are given in table 5. These profiles have significantly lower mean slopes, ranging from 0.0001 to 0.0007, indicating that general profile relief as well as profile shape is significantly affected by volume of sediment load.

Regression results for runs with varying discharge downstream and little or no spatial change in caliber indicate transport-threshold profiles exactly fit by power functions, as was the case in runs 19 and 20 under conditions of higher sediment discharge. So in the case of

TABLE 5

STANDARD DEVIATIONS FOR REGRESSION CURVES BASED ON THREE SIMPLE FUNCTIONS, PROFILES RESULTING FROM ADDITIONAL, SIMPLIFIED RUNS

Run No.	Sediment Caliber (mm)	Profile Concavity C_A	Standard Deviations, σ (m) for fit of simple functions		
			Exponential	Power	Log.
(Q const., Q_s const.)					
17	$d_o = .4, d \propto e^{-.003x}$	.020	.0102[a]	.0374	.115
18	$d = 15, d \propto e^{-.02x}$	.189	.306	.316	.231[a]
($Q \propto x^{1.2}$, Q_s const.)					
19	$d = .4$	.344	1.82	.0097[b]	.150
20	$d = 6.0$	.295	1.48	.0159[a]	.375
(Q const., $Q_s \propto x^{1.2}$)					
21	$d = .4$	−.212	.580	.0097[b]	.964
22	$d = 6.0$	−.144	.277	.0074[b]	.481
($Q \propto x^{1.7}$, $Q_s \rightarrow 0$					
23	$d_o = .4, d \propto e^{-.003x}$	.186	.0728	.0002[b]	.0470
24	$d_o = 15, d \propto e^{-.02x}$	.586	.670[a]	1.73	3.26
25	$d = .4$	.180	.0734	.0006[b]	.0484
26	$d = 6.0$	.252	.485	.0041[b]	.179
($Q \propto x^{1.2}$, $Q_s \rightarrow 0$)					
27	$d_o = .4, d \propto e^{-.003x}$	.159	.0527	.0001[b]	.0373
28	$d_o = 15, d \propto e^{-.02x}$	.539	.382[a]	1.10	1.86
29	$d = .4$	.153	.0532	.0006[b]	.0386
30	$d = 6.0$	.180	.274	.0015[b]	.152
(Q const., $Q_s \rightarrow 0$)					
31	$d_o = .4, d \propto e^{-.003x}$	.010	.0007[b]	.0011[b]	.0220
32	$d_o = 15, d \propto e^{-.02x}$	.422	.0894[a]	.873	.861

[a] Best fit to profile, of three tested functions.
[b] "Exact" fit to profile, with error less than that potentially imposed by model numerical solution method.

sole variation of water discharge downstream, the complex transport equations as well as their complex transport-initiation functions yield simple profiles of equilibrium following the form of the discharge change. Transport-threshold profiles developed with caliber and discharge changing significantly downstream (runs 24 and 28) do not distinctively follow any simple function but are fit most closely by the exponential form. Finally, runs with sole change in caliber downstream yield transport-threshold profiles fit well by exponential curves, though not exactly so in the case of strongest size variation. Although the fit is very precise in run 31, its almost-straight profile does not recommend it as a test case. Whether we consider profiles developed under criteria of transport inception or of significant sediment transport, it appears that the assumption of a simple relation between sediment caliber and channel slope (Shulits 1941) is at best an approximation.

SENSITIVITY ANALYSIS

To explore this mathematical model we have had to adopt specific, constant values for some variables potentially affecting profile form, and specific ranges of value for others. It is apparent that streams with vastly different controls (including vastly larger size, considering the change in the length-area relationship) need to be investigated separately. But might seemingly insignificant changes in the values of particular controls produce significant modification of the present results? To analyze sensitivity of results to individual inputs, a typical model run is recalculated many times with one control value altered in each case, and the resulting profiles are analyzed for changes in general slope, concavity, and relative fit of the three simple functions.

Run number 9 is well suited for application of such a test; it is a moderate example of the set of general runs in terms of concavity, overall slope (0.0014), and relative fit by ex-

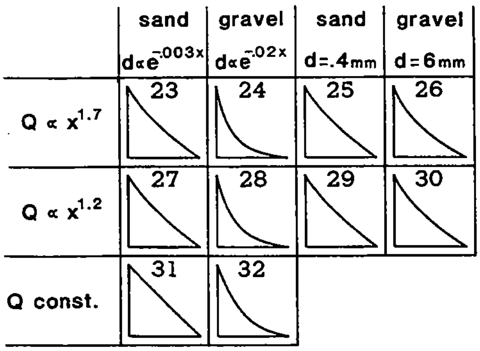

FIG. 7.—Stream bed profile forms calculated for transport-threshold conditions, model runs 23–32. Imposed relation of flow discharge Q to distance downstream x is shown on the vertical axis of the matrix, and downstream variation of sediment diameter d is given on the horizontal axis.

ponential, logarithmic, and power functions. The profile is modeled using Yang's transport equation, which is supported by a larger data base than Engelund's equation.

Controls Taken as Constants.—This first set of sensitivity analyses treat hydraulic roughness, sediment density, water temperature, and the exponent Z relating channel width to discharge, all factors taken as simple constants in the present study. For each analysis, two parallel model runs employ values of the investigated variable that not only bracket the original value used in run 9, but also encompass a large portion of the values found in nature. In all cases but the last, over this value range a power function best fits the stream profile best fit by a power function, by an order of magnitude better than other functions tested.

Taking 0.02 and 0.04 as alternative values of Manning's n for hydraulic roughness does not allow for representation of some high-resistance channels found in nature, but those numbers do cover a significant range of natural conditions. Variation over this range of roughness produces significant change in slope only, giving a total increase of 32% with increasing value of n. Although profile relief is affected, profile form remains constant. A similar effect is seen when the changing variable is sediment density. Over a specific gravity range of 2.3 (gypsum) to 3.2 (olivine), there is no significant alteration in profile form, but only about 20% difference in gradient, with higher slopes balancing higher density. Considering values of mean water temperature at high flow that fit in a range from 5° to 25°C, we find an insignificant change in slope at 2%, and a 6% decrease in concavity with an increase in temperature. Although water viscosity at 5°C is about 50% greater than that at 25°C, the effect on profile form cannot be considered more than a curiosity.

The sensitivity test values of the width exponent Z are set at 0.3 and 0.7, but it may be noted that there are cases (e.g., Osterkamp and Hedman 1982) where this exponent approximates 1.0, making width directly proportional to discharge. As Z increases from 0.3 to 0.7, overall slope decreases about 15% and concavity decreases about 65%. With W_{max} held constant at the downstream end, the exponent 0.7 amounts to a narrower channel upstream than does the lower exponent. Relatively then, both discharge per unit width and sediment load per unit width are increased upstream to an equivalent degree. But the results indicate that added transport capacity provided by increased Q/W more than accounts for the increase in Q_s/W, allowing somewhat lower slopes. Because this effect of added transport capacity relative to runs with lower exponent values decreases downstream, a less concave profile is developed.

For the range of width exponents given, the calculated profiles are fit by power functions with about one-tenth the error associated with the other functions. But the relative advantage of the power function, indicated by ratios of standard deviations of residuals, is about cut in half over this range, with the decrease in advantage following an increase in exponent value. One can extrapolate to understand this effect. As the exponent value goes to one, then Q/W becomes a constant value in the downstream direction, and if sediment discharge is proportional to discharge, then Q_s/W is also constant downstream. Cases satisfying these constraints have already been calculated in runs 17 and 18 (table 5), for which the power function approximation is not superior. High rates of width increase with spatial increase in discharge will diminish or eliminate the influences of discharge and sediment discharge on profile form. Although the mathematical experiments in this study do not provide support for the idea of a distinctive, simple profile form for stream reaches where sediment size variation is the only active control, the results do imply cases, other than the commonly noted constant-discharge streams, where such exclusive control on profile form by sediment factors may be in effect.

Main Control Variables.—Flow discharge, sediment discharge, and sediment size are all variables whose natural values span many orders of magnitude, and no simple sensitivity analysis addressing such ranges should be attempted. However, some information may be gained by this technique regarding model sensitivity to the general values of these controlling variables, doubling and halving the values of Q_{max}, Q_{sm}, and d_o to allow for model response over a four-fold range in each case.

Changes in overall gradient resulting from isolated change in these three variables are different in degree, with 114% slope change accompanying four-fold change in discharge, 88% slope change resulting from an equivalent change in sediment load, and 58% slope change across a four-fold range in grain size. The only significant change in profile concavity is brought about by varying sediment size, decreasing 14% over the range as general grain size increases. The relative advantage of the power function in matching the profile form is maintained in these tests at about 10-fold.

DYNAMIC MODELING OF APPROACH TO GRADE

To perform numerical experiments on stream approach to grade, we must model progressive changes in disequilibrium stream profiles over time. This is done by solving the set of equations (1–3) in their full form, using an implicit, time-weighted, finite-difference method given by Fread (1978). The four-point, central difference approximations used in this method are replaced by six-point approximations in equation (3) to allow greater flexibility in choice of sediment-related boundary conditions (Snow 1983). To solve the finite-difference forms of the equations, a generalized Newton iteration procedure (Amien and Fang 1970) is applied. This solution method has been demonstrated to provide economical solutions on computer for modeling of river profile adjustments over long time spans of interest to geomorphologists (Snow 1984).

With three equations insufficient to solve for four dependent variables, channel width is taken as an unchanging value through time. This is in accord with the observation by Wolman (1955) that aggrading and degrading streams are indistinguishable on the basis of hydraulic geometry from streams at grade. An assumption implicit in this stream model is the requirement that all channel slope adjustments be produced by erosion and deposition while channel sinuosity remains constant. If a model allowing stream segments to expand or contract over time were to be developed, it would require a mathematical expression relating sinuosity changes to erosive force, bank material, and other factors, and such an expression is presently unavailable. The stream is assumed to be eroding in alluvial fill, so that channel degradation is a matter of transport, not abrasion.

We have taken the set of external controls appropriate to steady-state run number 1 as input to the dynamic model, performing three numerical experiments of profile approach to grade from arbitrary, initial stream profiles (fig. 8). The initial profile for run A is a straight line, determining that in erosion to a concave form the thickest column of sediment must be removed from the central region of the stream course. Run B begins with a convex profile reminiscent of an uplifted land surface. In this case, most of the sediment to be removed is in the lower reaches of the stream. The initial profile for run C is simply the graded profile from run 1 with all slopes increased by a factor of 1.5, giving a profile equivalent in concavity to the profile to be finally developed. The majority of sediment must be removed from the upstream reaches.

These model runs can only be taken as a first approximation to actual histories of major downcutting and aggradation in river valleys. The model does not include valley response such as increased sediment inflow resulting from downcutting. But our purpose here is different from that; our interest is in the characteristics of stream profiles as they near the graded condition. The last few meters of erosion are unlikely to provoke any types of basin response that have not already become prevailing conditions as results of rapid, initial downcutting.

Times given for profile adjustment in these

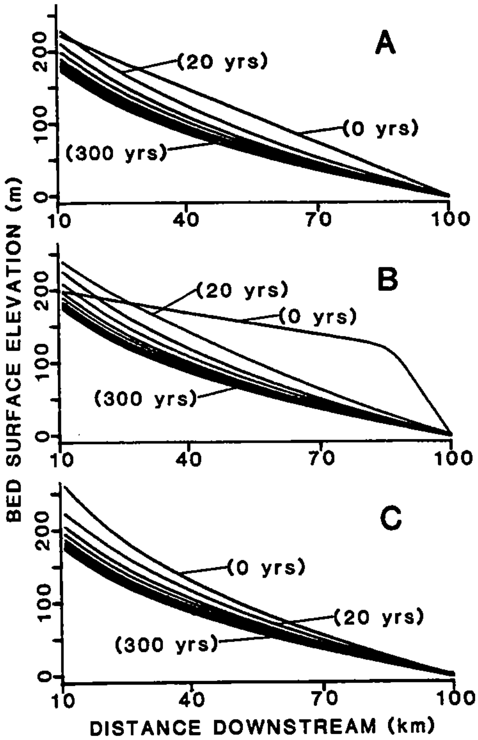

FIG. 8.—Initial, intermediate, and final longitudinal profiles for three dynamic model runs of profile approach to grade. Each space between profiles represents 20 yrs of model time and an unknown, larger number of years in real time. The initial profile for run A is a straight line, that for run B is convex, and that for run C has a concavity equal to that of the final, graded profile. Signs of initial conditions are rapidly obliterated, and intermediate profiles quickly come to reflect the form of the graded profile.

model runs are underestimated not only because removal of valley sediments outside the channel is not accounted for, but also because the "channel-forming flow" given for discharge far exceeds the sediment-transport capacity of flows occurring most of the time. The "time to grade" of about 300 years given here can only be taken as a very conservative lower limit for relaxation times of real streams with similar conditions. Because of the need to use some sort of time notation in discussing results, we will speak in terms of "model years."

The diversity of initial profiles tests the degree to which such initial characteristics are retained in the profiles developing to grade. Even with the extreme differences in required amounts and locations of erosion, the experiments indicate that signs of the initial conditions are quickly obliterated. With tens of meters of erosion remaining to be accomplished, the intermediate profiles are very similar in form to the final profile, and in all three runs the subsequent profile lowering proceeds in the same asymptotic manner.

The only difference noted between profile forms for streams approaching grade and those at grade is a slightly lower concavity for profiles approaching grade, about 1% less at a model time of 150 yrs. This difference is useless for analysis of actual stream profiles, but still ought to be explained. While a stream erodes, an extra sediment discharge originating from the channel bed is added to the externally imposed sediment load, and accumulates downstream. As we have seen, a greater downstream rate of increase in overall sediment load requires a less concave equilibrium profile. Here we see a similar effect on the profile approaching equilibrium. This effect most distinctively shows in the evolution of the stream profile in run C: the initial profile is equal in concavity to the final profile, but the intermediate profile at 20 model yrs is about 10% less concave, and subsequent concavity values increase again, asymptotically approaching the equilibrium value.

CONCLUSIONS

Based on the results of these numerical experiments in the calculation of ideal river profile forms, we can suggest two general reasons for the variety of types of mathematical curves that fit equilibrium stream profiles. First, if origin values are not specified, then exponential, logarithmic, and power function curves can all be fit quite closely to stream profiles such as those modeled here. Perception of which curve form is the best analog to particular profiles in the field is a function not only of profile form but also of the set of curve forms tested and the constraints placed on each function by specifications of curve origin.

Second, the set of profiles modeled shows variation in relative advantage of different curve types as profile analogs. Stream systems characterized primarily by downstream increases in fluid and sediment discharge develop profiles distinctively of power-function form. If there is also a strong decrease in sediment caliber downstream, then profile form is not matched well by any of the simple functions, but a logarithmic curve gives the best approximation. Profiles developed at low rates of sediment transport with sediment size decreasing downstream and other major controls held constant most closely approach pure exponential curves. If the increase of width downstream is nearly proportional to the increase in discharge, then the influence of discharge variation on profile form is negated, and sediment-related factors shape the profile.

The model results relating to profile form can be generalized over a wide range of values for channel roughness, sediment density, and water temperature. Doubling or halving the prevailing values of discharge, sediment load, and sediment size significantly affects slope but does not produce major changes in profile shape, though some decrease in concavity accompanies an increase in general sediment size. There is reason to believe that the model presented here cannot adequately treat streams with much larger drainage basins because the relation of mainstream length to drainage area changes at larger scales. The model is limited in applicability to ranges of sediment caliber values appropriate to the sediment transport equations employed. In light of changes in calculated profile form between sand and fine gravel sizes, profiles developed under otherwise similar conditions may show significant changes not only in slope but in shape over the full range of river bed material sizes.

Rivers eroding their channels and approaching to the graded state develop profiles very similar to graded profiles quite early in the time between initiation of erosion and attainment of equilibrium. Conditions of grade and approach to grade cannot be distinguished on the basis of profile form; other stream characteristics must be examined. Although streams of the type modeled here may well have relaxation times measured in thousands or tens of thousands of years and, therefore, may be in a continual state of adjustment to long-term changes in superficial conditions, the characteristics of profile form that we have examined for the ideal graded case are likely to remain in effect.

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