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Predictability and Chaos in Quantitative Dynamic Stratigraphy

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Abstract

Models of quantitative dynamic stratigraphy are usually nonlinear equations representing forced, dissipative systems, and as such they are susceptible to a rich mathematical behavior only recently recognized. Even a simple nonlinear dynamical system such as the Lorenz equations describing Rayleigh-Benard convection, used here as an example, contains periodic, slightly aperiodic, and seemingly random solutions called chaotic, depending upon the Rayleigh number. Systems of this type display a sensitive dependence upon initial conditions making prediction in its present sense impossible. Some periodicities that arise are likely to be explained by external causes when in fact, they are due to nonlinear coupling. In a positive light, these nonlinear dynamics may explain some of the complexities in the stratigraphic record.

Introduction

Over the last two decades a quiet revolution has occurred in the science and mathematics of nonlinear dynamical systems. What was once a backwater topic of research, for the most part ignored by physicists after Poincaré, now has its own journals, conferences, centers for nonlinear studies, and even its own toys, such as Space Balls. The reasons are several—the diminishing returns of particle physics, advances in computers and numerical analysis, for example—but two others seem especially noteworthy. first, we are now at the stage where the interesting problems are the more difficult nonlinear ones. To practice reductionist science with its linearized models is to throw out the baby with the bathwater. Thus an alternative scientific approach, termed "analysis by synthesis" by Hut and Sussman (1987), has arisen wherein one constructs and solves nonlinear mathematical models on the computer. Of a set of models, the model configuration that best accounts for the observations is assumed to be the correct one. This is the technique of quantitative dynamic stratigraphy (QDS) as presented elsewhere in this volume. Second, as scientists in as diverse fields as meteorology and population ecology began constructing nonlinear models, they discovered a rich mathematical behavior (see Crutchfield et al., 1986, and Stewart and Thompson, 1986, for reviews, and Gleick, 1987, for a popular account). Even the simplest of deterministic equations generated periodic, slightly aperiodic, and random solutions, the latter being called chaos. This rich behavior drew the interest of mathematicians, making the study of nonlinear dynamical systems fashionable once again. The result has been a deeper understanding of such enigmas as the transition to turbulence in fluids (Feigenbaum, 1980; Hofstadter, 1981).

There is a certain irony in the revolution however. The possibility of chaotic behavior in even simple nonlinear deterministic systems makes analysis by synthesis all the more difficult. Over a certain range of initial conditions, the solutions may be well behaved, settling down to a fixed point or simple orbit in the state space. In other, *a priori* unknowable ranges, the solutions may be chaotic. And, in either case, small differences in the initial conditions may produce great differences in the solutions, thus magnifying small errors in the initial conditions. Prediction becomes impossible.

Two questions arise then—how do we recognize chaos, and are quantitative dynamic stratigraphy models susceptible to it? If the answer to the latter is yes, an additional question follows—what are the stratigraphic implications of a chaotic solution? The remainder of this article addresses these questions, although no straightforward answers are presented. A review of the Lorenz model of Rayleigh-Benard convection provides an analogue for recognizing chaos and I will attempt to summarize the few known necessary conditions for chaotic behavior of a system. Some comments on stratigraphic implications close the discussion.

An Example of Nonlinear Behavior and Chaos

Classical Rayleigh-Benard convection serves as an ideal example of nonlinear behavior, because it is simple enough to allow for intuitive understanding and because it is quite well studied. Rayleigh-Benard convection is one possible mode of fluid circulation deep within sedimentary basins where it could contribute to the origin of diagenetic pressure seals of gas reservoirs.

When a fluid is heated uniformly from below and cooled uniformly from above, heat is first transported vertically by conduction with no apparent fluid motion. After some time, and for temperature differences above a critical minimum, cylindrical rolls develop, convecting heat by fluid transport. As demonstrated in Shirer (1987), the smallest effective model of these system states through time is the two-dimensional equation set of Lorenz (1963),

$$dx/dt^* = -px + py$$

$$dy/dt^* = -xz + rx - y$$

$$dz/dt^* = xy - bz$$

where x is proportional to the intensity of convective motion, y is proportional to the temperature difference between ascending and descending currents, z is proportional to the distortion of the vertical temperature profile from linearity, and t^* is dimensionless time. The coefficient p is the *Prandtl number*, taken as 10 in the original calculations; r is the normalized *Rayleigh number* (equal to 24.7 at the onset of steady convection), taken to be 28; and b is a constant equal to 8/3. For an initial condition, Lorenz chose (0,1,0), a slight departure from the state of no convection.

Our intuition tells us that at this slightly supercritical normalized Rayleigh number, r, fluid flow should commence at t* > 0 and evolve into a steady state convection. Lorenz found otherwise. The temporal behavior of solutions to this small system of equations is complicated for this and certain other r in the range 24.7 < r < 215. This is illustrated by plotting y, the difference in fluid temperatures on the rising and falling limbs of a cell (Fig. 1). It grows with time up to a t* of about 30 when warm fluid is at the top of the cell. Then y decreases and by t* = 50 changes sign, signifying that the overly vigorous flow of the cell has caused the warm fluid originally at the bottom of the cell at t* = 0 to continue over the top of the cell and descend; likewise the cold fluid ascends. The resulting buoyant forces cause the motion to cease and reverse direction at t* = 60. For 85 < t* < 1650 the fluid motion matches our intuition in that it is quasi-steady around one solution with a fixed mean value of y (also x and z). It oscillates however, and the oscillations increase in amplitude until t* = 1650 after which the motion is quite irregular. The motion is sometimes clockwise and sometimes counterclockwise with no apparent long-term periodicity.

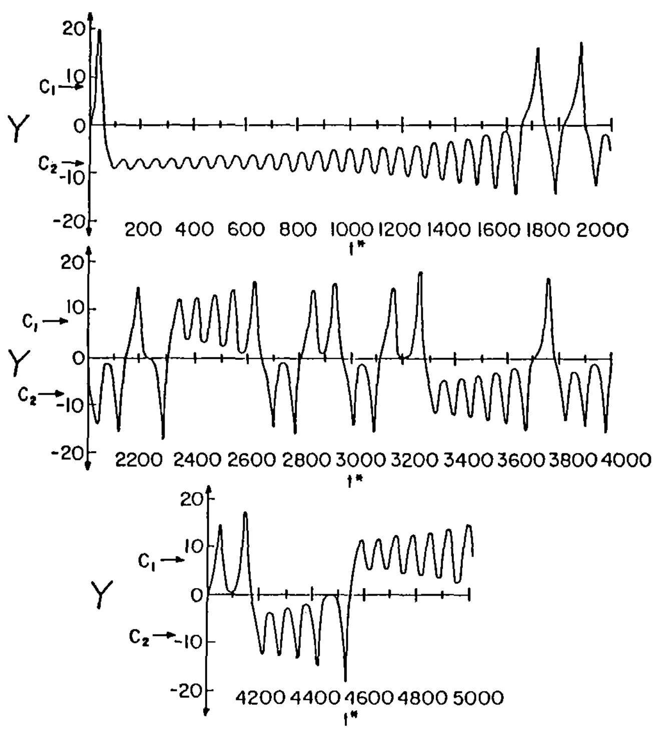

Figure 1. Times series solution of the Lorenz convection model for p = 10, b = 8/3, and r = 28. Variable y is proportional to the difference in fluid temperatures on the rising and falling limbs of the convection cell and t* is dimensionless time. At t* = 1650 the motion becomes chaotic (modified from Nese, 1985).

Is the motion chaotic? To answer this question, consider a more revealing graphical representation—a plot of the solutions in their state space, an abstract construct whose coordinates are the dependent variables of the system. A system that proceeds from some initial condition to a steady state solution would be represented by a trajectory from an initial point to a single steady state point. Because many systems end up at the same steady state solution regardless of initial conditions, a steady state point is said to attract nearby trajectories or *orbits* and is called an *attractor*. There may be many attractors in a state space, each with its own *basin of attraction*. Other systems may not come to rest in the long term; rather they may cycle periodically thorough a sequence of states in a *periodic orbit*. The associated attractors are called *limit cycles* (see Crutchfield et al., 1986; May, 1976; and Stewart and Thompson, 1986).

The graph of the Lorenz equations in state space (Fig. 2) was constructed by solving the equations at timesteps so infinitesimal as to produce a line. The trajectory loops around one stationary solution and then another, returning near to itself but never duplicating an individual orbit. Solutions such as these are called *chaotic*. They are random in the sense that no predictions about future states can be made, yet they arise from a completely deterministic system. Attractors of this type are called *chaotic* or *strange* attractors (see Devaney, 1986, p. 50, for formal definitions). Interestingly, they have a fractal nature; an infinite number of points show a self-similar detail at all levels of magnification.

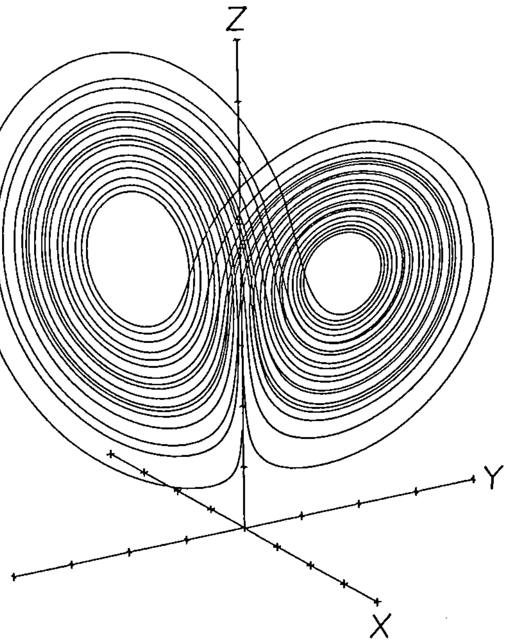

Figure 2. Plot of solutions to the Lorenz convection model in state space for the same conditions as Figure 1 but starting after $t^* = 1650$. The trajectory loops around two stationary solutions but never repeats itself, thereby defining a strange attractor (modified from Nese, 1985).

To grasp just how chaotic this behavior is, consider the experiment performed by Crutchfield et al. (1986) illustrated in Figure 3. Solutions of the equations are shown in state space at selected times for each of 10,000 initial conditions, so close together they appear as one dot at t* = 0. The solutions spread out through time to cover the entire attractor, dramatically illustrating sensitive dependence on initial conditions and the unpredictability of future states. This sensitivity has become known as the butterfly effect, from Lorenz's (1979) address entitled, "Predictability: Does the flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?"

To summarize, the above example illustrates that some forced nonconservative hydrodynamical systems may exhibit quasi-periodic behavior over the short term with no periodicity in the forcing. Over the longer term they may show chaotic behavior, depending upon the magnitude of the coefficients. The chaos is unpredictable, sensitive to initial conditions, yet bounded, and recurrent, producing a fractal geometry.

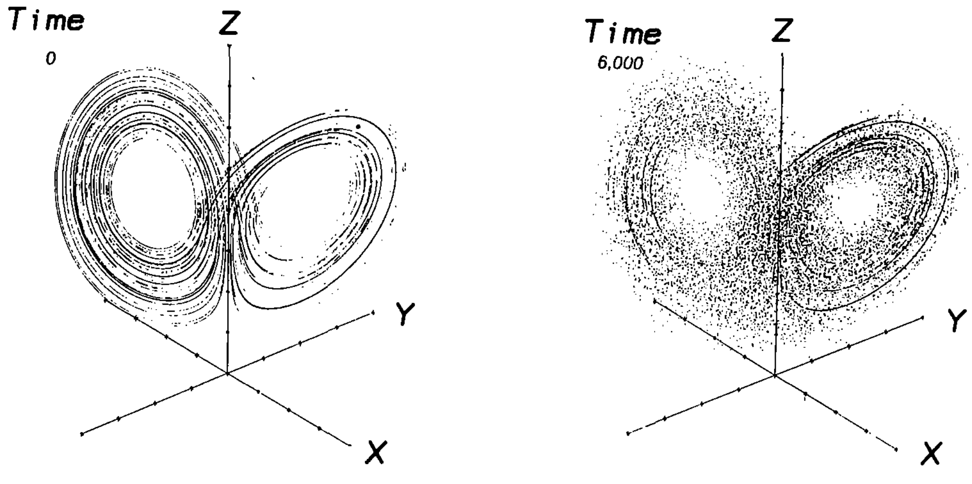

Figure 3. The Lorenz strange attractor of Figure 2 with additional solutions overlaid to illustrate sensitive dependence to initial conditions. At t* = 0 a point represents 10,000 initial conditions that lead to the 10,000 solutions presented at t* = 6,000 (modified from Crutchfield et al., 1986).

Are QDS Models Susceptible to Chaos?

Although there is no formal mathematical answer to this question, there are some general guidelines we can use. These guidelines, which will be considered in turn, are:

1. Classification of the model's system of equations with respect to physical type, dimension, degree of coupling among equations, and degree of nonlinearity;
2. Values of the Lyapunov dimension and correlation exponent; and
3. Sensitivity to initial conditions.

Classification

A QDS model can be classified first according to physical type. By this is meant whether it is an open or closed organizational, kinematic, or dynamical model, and if the latter, whether it is forced and whether it is dissipative. Most of the QDS models presented in this volume are open and dynamical in that they receive mass from outside and go beyond the geometrical relations of the kinematic model to include evolution of the state variables with forces also considered. Most are forced in that they are fed energy by a boundary condition, and most are dissipative in that they lose energy through friction. It is now understood that open, forced and dissipative dynamical systems such as these are susceptible to chaotic solutions (Shirer, 1987). This arises because of the competition between the forcing and dissipative processes. Also, it has been argued on thermodynamic grounds that systems closed to their environment with respect to mass transport should not exhibit instability (Feinberg, 1980).

Dimension or number of degrees of freedom seems to be an important consideration for chaotic behavior. The Lorenz attractor disappears in a three-dimensional convection model (although new chaotic attractors appear), probably because turbulence is three-dimensional and Lorenz's two-dimensional model is hunting for a stable solution that only is available with another degree of freedom (Shirer, 1987). This raises the issue, long debated in population ecology, of whether more complex systems are more stable or less stable than simpler systems. In this usage, complex means both more variables and greater degree of coupling among variables. The general conclusion from qualitative stability analysis of partially specified systems is that progressively more complex systems are likely to be progressively less stable (Levins, 1974). However, Shaw (1987, p. 1653) concluded the opposite: "Computer experiments show that the coupling together of complex systems often increases . . . the degree of order in the composite system."

Finally, one might suspect that the degree of nonlinearity may determine whether a system exhibits a chaotic attractor. No general rules seem to exist on the subject, however (H. N. Shirer, personal communication, 1988).

Lyapunov Dimension and Correlation Exponent

The Lyapunov Dimension and Correlation Exponent are thought to measure the number of dimensions necessary to specify the region of the attractor in the state space. For example, if there are $N = 3$ equations in a dynamical system, then the state space has three axes corresponding to the three state variables, and the largest dimension possible for an attractor is 3 or generally, N. This would be a volume in the state space that attracts or traps trajectories orbiting near it. Similarly, the point and limit cycle attractors mentioned earlier would have dimensions of 0 and 1, respectively. This information becomes useful in the present context because the dimensions of strange attractors are usually nonintegers, a reflection of the folded, fractal structure of the chaotic solution sets. Thus, determining whether a model will exhibit chaotic behavior reduces to determining the dimensions of its attractors.

The Lyapunov dimension was defined by Kaplan and Yorke (1979) as a function of the Lyapunov exponents of an attractor. In the interests of brevity and because the correlation exponent is easier to calculate, the Lyapunov dimension will not be discussed further here; see Nese (1987) for details.

The correlation exponent v, was defined by Grassberger and Procaccia (1983a,b) as a measure of the local structure of an attractor. It is conjectured to be related to the Lyapunov dimension and can be calculated from a time series of one component of the dynamical system. Let Y_j,j = 1,2,...n, be n points on an attractor residing in N-dimensional state space. The points may be obtained from a QDS model, for example, as a times series $Y_j = Y(t+jT)$ of a dependent variable, where T is a fixed time increment. A point Y_k is selected and all the distances $\|Y_j-Y_k\|$ of this point from the remaining n-1 points are calculated. This procedure is repeated for all the Y_k points on the attractor and a correlation integral C(L) is computed as

$$C(L) = \lim_{n \to \infty} \frac{1}{n^2} \sum_{\substack{j,k=1 \\ \text{when } j \neq k}}^{n} H(L - \| Y_j - Y_k \|)$$

where H is the Heaviside function (if $L - \|Y_j - Y_k\| > 0$, then H = 1, otherwise, H = 0) and L is a fixed distance measured from Y_k. This is equivalent to calculating the density of points on the attractor within a range of distances L from Y_k, and then finding the average of this density over all values of k. In general one expects that

$$C(L) \propto L^v$$

where v, the correlation exponent should be 1 if the attractor is a line, 2 if a surface and so on up to N, the dimension of the state space. In the latter case the data points are totally uncorrelated, i.e., random. Operationally, v is determined by finding the slope of the line when ln[C(L)] is plotted against ln[L].

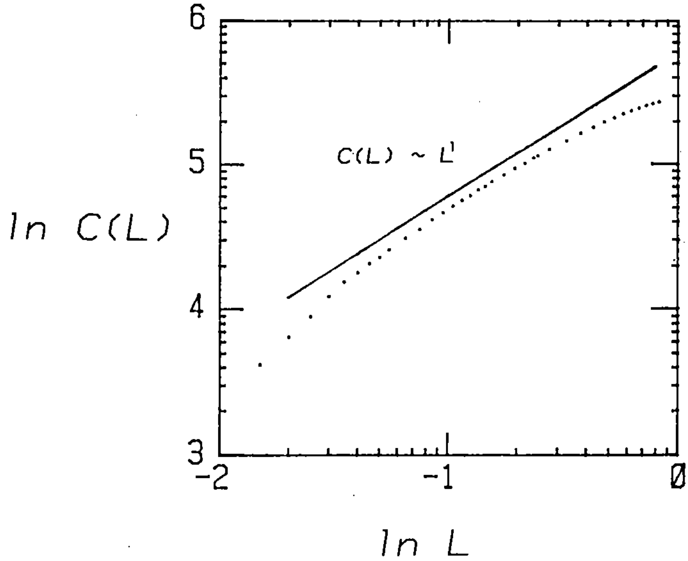

Figure 4. Graph of the correlation integral versus L computed using 4,000 points on the periodic attractor of the Lorenz system when p = 10, b = 8/3, and r = 300. The slope of the points is approximately one, indicating the attractor is a line, and therefore nonchaotic.

As an example, consider a plot of this type for the Lorenz system (Fig. 4) when the normalized Rayleigh number r, is 300 (Nese, 1985). At this r the attractor is a stable limit cycle, meaning that the solutions have settled down to stable oscillations and the graph of the state space is a loop. By the above reasoning we expect v to be 1, a value closely approximated by the slope of the data points in Figure 4. For the case discussed above when r = 28 and the attractor is chaotic, Grassberger and Procaccia (1983a,b) calculated a v of 2.05 + 0.01, a fractal dimension as expected. Thus, it may be possible to examine a QDS model's output for chaotic behavior by examination of its correlation exponent.

Sensitivity to Initial Conditions

Probably the most straightforward method for determining a model's susceptibility to chaotic behavior is to test it for sensitivity to initial conditions. A chaotic attractor is strongly suggested if for similar (but not identical) initial conditions the solutions show early similar behavior that diverges with time.

It appears then, that as a class, QDS models could be susceptible to chaotic behavior, especially when integrated over long time periods. Solutions can be inspected for chaos, however, and this possibility should be considered along with other more common explanations such as numerical instability.

Implications of Chaos to Stratigraphy

There are several implications of chaotic solutions to QDS models. First, they make prediction difficult. Without chaos we expect the final configuration of sedimentary facies, for example, to be only weakly affected by slight changes in the initial conditions. With chaos, any configuration within the region of the attractor is possible. Second, as Shaw (1987) pointed out, there need be no unique causative periodic forcing required to explain apparent periodicities in the rock record. They can arise from the nonlinear coupling as sets of interacting resonances, much as in the Lorenz model (Fig. 2). The seemingly periodic repetitive successions of lithologies, currently interpreted as a Milankovitch signal, should be examined with this in mind. Finally, and in a more positive vein, certain complexities of the stratigraphic record may now have an explanation in chaos theory.

Conclusions

This paper has attempted to define typical behaviors of nonlinear dynamical systems, particularly chaos. It has explored the extent to which quantitative dynamic stratigraphic models may be susceptible to chaos, and has suggested some implications for stratigraphy. There is every reason to believe that some QDS models will contain chaotic attractors in their state spaces; indeed, this situation may be necessary if we are ever to explain complexities of the stratigraphic record.

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