Qualitative stability analysis of geologic systems, with an example from river hydraulic geometry

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ABSTRACT

Many geologic systems are at present only partially specified, in that the variables and positive and negative feedback loops are known but the exact functional relationships among variables are not. It is still possible to describe the response of these systems to a perturbation by analyzing the eigenvalues derived from the coefficient matrix of the system equations evaluated near an equilibrium point. The method predicts that a simple model of stream at-a-station hydraulic geometry is metastable provided the relative rates of change of friction factor, hydraulic radius, and slope are large, intermediate, and small, respectively.

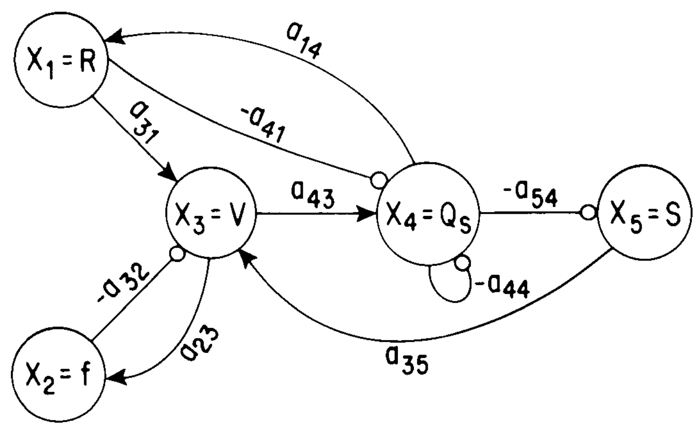

Figure 1. Signed, directed graph of simplified stream-channel system. Arrows and open circles indicate positive and negative feedback loops, respectively. The a_{ij} represent partial rates of change of variable *i* with variable *j*.

INTRODUCTION

Geologic systems are usually rich in variables and interconnectedness and consequently are usually quite intractable. They often are governed by sets of nonlinear differential equations that are notoriously difficult to solve, especially as the number of elements or connections increases beyond a few. Worse, many have such poorly known functional relationships among variables that quantitative solutions are not possible at present. A class of this kind, where the types of feedback among variables (positive or negative) are known but the exact functional relationships are not (for example, Fig. 1) is called *partially specified* (Levins, 1974). Chorley and Kennedy (1971) gave numerous geologic examples.

Two questions of interest concerning partially specified geologic systems are, What is the response of a system when perturbed? and How do the speed and direction of that response depend on the strengths of the individual links in the feedback loops, the simplicity of organization, and the various relaxation times involved? For example, we could ask, How does a reach of a stream change its slope, channel shape, and other variables in response to dredging? Does it return to the initial state, and if so, how long does it take? If not, does it find a new equilibrium configuration or does it increasingly deviate from its initial state? These three responses characterize stable, metastable, and unstable systems, respectively.

This paper presents a method drawn from stability theory for dynamical systems (compare Braun, 1978; Porter, 1967; Levins, 1974) to answer such questions for this general class of systems. The method tests for stability characteristics around a system's equilibrium points and for the impact that a varying element has on the levels of its coexisting elements. If a system's response to disruption of an equilibrium point is known, say empirically, then the model defining the system's dominant variables and feedback loops can be tested for accuracy. Conversely, if the model is accepted as accurate, the response of each prototype variable to a perturbation of another can be explored along with the prototype's stability as a whole. In either case, the analysis specifies the ranking of the rates of change of one variable with another that is necessary for system stability.

THEORY

Consider a proposed geologic system of *n* variables, $\vec{X}$, whose levels vary with time, *t*, as functions, $\vec{F}$, of each other. That is,

$$\frac{d\vec{X}}{dt} = \vec{F}(\vec{X}). \tag{1}$$

This formulation assumes that the system variables adjust to arrive at states acceptable to each acting in the conditions given by the other variables. The $\vec{F}$ may be nonlinear but may not depend explicitly on *t*. As will be shown later, for this analysis it is not necessary to know the form of the $\vec{F}$, only whether an X_i is increased, decreased, or unaffected by a change in another X_j. For example, Figure 1 shows a model of a five-variable system in which, following Levins (1974), positive and negative feedbacks ($\vec{F}$) among variables are denoted by an arrow and an open circle, respectively. If there is no connecting line, the implication is that variables do not *directly* affect each other. Equation 1 written for X_3 would be

$$\frac{dX_3}{dt} = F_3\,(X_1, X_2, X_5).$$

How does this system respond to a perturbation of the $\vec{X}$? We can answer this by exploring the conditions for stability of equation 1 around its equilibrium points.

Following Lotka (1956) and Braun (1978, p. 365), let $\vec{C}$ be an equilibrium point of $\vec{X}$ and let $\vec{F}(\vec{X})$ have two continuous partial derivatives with respect to each of its variables. Then a deviation of $\vec{X}$ from equilibrium is

$$\vec{x} = \vec{X} - \vec{C}. \tag{2}$$

Substituting equation 2 in equation 1,

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x} + \vec{C}). \tag{3}$$

By Taylor's Theorem, the right hand side of equation 3 can be written as (Braun, 1978)

$$\vec{F}(\vec{C}+\vec{x}) = \vec{F}(\vec{C}) + \mathrm{A}\vec{x} + \vec{g}(\vec{x}), \tag{4}$$

where $\vec{g}(\vec{x})$ is a vector of polynomials with terms of two or higher order, each small compared to $\vec{x}$, and which vanish at $\vec{x} = 0$ and

$$\mathrm{A} = \begin{pmatrix} \frac{\partial F_1}{\partial X_1}(\vec{C}) & \cdots\cdots & \frac{\partial F_1}{\partial X_n}(\vec{C}) \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial X_1}(\vec{C}) & \cdots\cdots & \frac{\partial F_n}{\partial X_n}(\vec{C}) \end{pmatrix}. \tag{5}$$

Because $g(\vec{x})$ is very small with respect to A $\vec{x}$ if $\vec{x}$ is small, it is mathematically sufficient to explore the linear ordinary differential equation system

$$\frac{d\vec{x}}{dt} = \mathrm{A}\vec{x} \tag{6}$$

for stability. Every solution to equation 6 is of the form

$$\vec{x}(t) = \vec{v}\vec{C}\exp[\vec{\lambda}t], \tag{7}$$

where $\vec{v}$ are eigenvectors of **A** corresponding to eigenvalues, $\vec{\lambda}$. The stability of equation 1 is determined by whether the real parts of the eigenvalues, $\vec{\lambda}$, of **A** are each greater than, equal to, or less than zero. This follows from equation 7, which shows that if all $\vec{\lambda} < 0$, all $\vec{x}$ approach zero as t approaches infinity. Therefore, from equation 2, all $\vec{X}$ approach $\vec{C}$, their equilibrium values. That is to say, the system is asymptotically stable near an equilibrium solution, $\vec{C}$. By the same reasoning, if at least one eigenvalue of A has a positive real part, the system is unstable and the values of the variables increasingly deviate from their equilibrium values with time. If at least one eigenvalue has zero real part with all other eigenvalues less than zero, the stability of the system can be determined for the nonlinear case only by assuming small perturbations. In that case and for any linear case, the system is metastable if there is one eigenvalue equal to zero or if there are k linearly independent eigenvectors for each of k multiple eigenvalues with zero real parts. Otherwise the system is unstable (Braun, 1978, p. 354).

Now note that the terms in A (equation 5) are, by definition, the a_{ij} of Figure 1. Thus, the signs of $\vec{\lambda}$ are determined by the a_{ij} of the system matrix, which themselves give the type (negative or positive) and amount (strong or weak) of feedback between variables i and j. In most cases, a system's stability can be specified with only a knowledge of the signs of the a_{ij}, determined, say from statistically significant correlation coefficients among variables. Otherwise, at least the relative orders of magnitude of the coefficients must be known. Analyzing the behavior of a system thus reduces to analyzing the roots (eigenvalues) of the characteristic polynomial of its a_{ij} matrix.

Because these characteristic equations frequently are higher order polynomials and hence are not easily solvable, criteria are needed for relating the roots of a polynomial to its coefficients. One useful criterion here is the Routh-Hurwitz criterion (Cesari, 1971), from which it can be deduced that all the roots of a real characteristic equation have negative parts if and only if all the coefficients, $\alpha_0, \alpha_1 \ldots \alpha_n$, of the characteristic polynomial are positive and, for the case when $n = 4$,

$$\alpha_1\alpha_2\alpha_3 - \alpha_3^2 - \alpha_4\alpha_1^2 > 0. \tag{8}$$

This will be used in the next section.

EXAMPLE

It is still not possible to predict the at-a-station hydraulic geometry of alluvial channels from known functional relationships among the important variables. As formulated by Hey (1978), there are nine variables and therefore nine degrees of freedom for channel adjustment: average velocity, V; hydraulic radius, R; channel slope, S; dune wavelength, Λ; dune height, Δ; wetted perimeter, P; maximum flow depth, d_m; channel sinuosity, p; and meander arc length, z; all are measured at bankfull flow. In addition, the boundary conditions are some given water discharge, upstream sediment load, bed and bank grain sizes, and valley slope. Because the exact functional relationships among these variables are not known but the types of association (either positive or negative) are, the system is *partially specified* and therefore amenable to the method.

Consider the simplified formulation of this system in Figure 1, where, for a stream reach with a given set of boundary conditions, X_1 = R, hydraulic radius; X_2 = f, Darcy-Weisbach friction factor; X_3 = V, mean stream velocity; $X_4 = Q_S$, bedload transport rate; and X_5 = S, bed slope. The original nine variables have been reduced to five by assuming a straight reach and letting R represent P and d_m, and f represent Δ and Λ. In this model, hydraulic radius is assumed to positively affect stream velocity (link a_{31} of Fig. 1) as in Manning's equation. Hydraulic radius negatively affects bedload discharge ($-a_{41}$) because sediment transport rates are inversely proportional to depth of flow, all other factors held constant. The friction factor negatively affects velocity ($-a_{32}$), and velocity positively affects the friction factor (a_{23}) by increasing bedform size at low to intermediate Froude numbers. Stream velocity positively affects bedload transport (a_{43}). Bedload transport is self-damped ($-a_{44}$) because the stream is capacity limited. In this formulation bedload-transport rate negatively affects slope ($-a_{54}$). Within a reach, an increase in Q_S implies that sediment is being scoured from the bed and banks. That scour *decreases* the local slope, as is well known from case studies of stream channelization. Likewise, a decrease in Q_S along a reach with resulting deposition increases the local slope until Q_S is just sufficient to transport the imposed load. This is not to be confused with relationships such as Lane's for an open fluvial system where Q_S and S are positively correlated. An increasing bedload transport rate in the reach is reasoned to increase the hydraulic radius (a_{14}) by deepening the channel or making it more nearly semihemispherical in cross section. As in the previous argument, increasing Q_S in a reach means increasing scour. For many streams (Park, 1977), this scour occurs more on the bed than on the banks, thus increasing R. Finally, as slope increases, so does mean stream velocity (a_{35}).

How will this nonlinear system respond to a perturbation of the $\bar{X}$? We can answer this in the sense defined previously by assuming that the stream is near an equilibrium point and the deviations from equilibrium are small. The system matrix is

$$A = \begin{pmatrix} 0 & 0 & 0 & a_{14} & 0 \\ 0 & 0 & a_{23} & 0 & 0 \\ a_{31} & -a_{32} & 0 & 0 & a_{35} \\ -a_{41} & 0 & a_{43} & -a_{44} & 0 \\ 0 & 0 & 0 & -a_{54} & 0 \end{pmatrix}. \quad (9)$$

That is, each link from variable X_j to X_i in Figure 1 represents the effect of X_j on X_i and corresponds to the matrix element a_{ij} in equations 5 and 9.

The characteristic equation of **A** is

$$\lambda^5 + a_{44}\lambda^4 + (a_{23}a_{32} + a_{14}a_{41})\lambda^3 + (a_{23}a_{32}a_{44} + a_{35}a_{43}a_{54} - a_{14}a_{31}a_{43})\lambda^2 + a_{14}a_{23}a_{32}a_{41}\lambda = 0 \quad (10)$$

or

$$\lambda^4 + a_{44}\lambda^3 + (a_{23}a_{32} + a_{14}a_{41})\lambda^2 + (a_{23}a_{32}a_{44} + a_{35}a_{43}a_{54} - a_{14}a_{31}a_{43})\lambda + a_{14}a_{23}a_{32}a_{41} = 0 \quad (11)$$

and $\lambda_5 = 0$.

Because equation 10 has one zero root, we can conclude that the system is at most metastable. That is to say, a stream may adjust its at-a-station hydraulic geometry to a perturbation by obtaining some new combination of values; for example, if its slope is increased a small amount, it will not return necessarily to its preperturbation value.

To determine if the system is unstable, we must determine if equation 11 has any zero or positive eigenvalues. Applying the Routh-Hurwitz criterion, $\lambda_{1,\,2,\,3,\,4}$ of equation 11 are less than zero if and only if:

$$a_{23}a_{32}a_{44} + a_{35}a_{43}a_{54} > a_{14}a_{31}a_{43}, \quad (12)$$

and from equation 8,

$$(a_{44}a_{23}a_{32} + a_{44}a_{41}a_{14})(a_{35}a_{43}a_{54} + a_{23}a_{32}a_{44} - a_{14}a_{31}a_{43}) > a_{44}^2a_{14}a_{23}a_{32}a_{41} + (a_{35}a_{43}a_{54} + a_{23}a_{32}a_{44} - a_{14}a_{31}a_{43})^2. \quad (13)$$

Equation 12 means that negative feedback in the f-V loop, Q_S self-loop, and V-Q_S-S loop must overpower the deviation-amplifying loop of Q_S-R-V. Equation 13 is more complicated. It is one of a class of stability requirements specifying that negative feedback coming from long loops cannot be too strong compared to negative feedback from shorter loops (Levins, 1974, p. 128). Here, equation 13 is generally satisfied if the f-V loop and Q_S-R loop are of unequal damping strengths.

Conditions 12 and 13 can be met most simply if the rates of change of variables with respect to each other in the f-V loop are largest, Q_S-R are intermediate, and V-Q_S-S loop are smallest. This ranking seems intuitively correct because we expect a stream to change its friction factor in (on the order of) hours, its hydraulic radius in years, and its slope in tens of years. Where the hydraulic radius can adjust at a faster rate than the friction factor, such as in glacial outwash streams, the model predicts instability, one manifestation of which may be braiding.

I know of no data documenting this sequence of adjustments of a natural closed fluvial system. Data are available, however, on the detailed response of a fluvial system to a change in a boundary condition. Andrews (1979) has documented the hydraulic adjustment of the East Fork River, Wyoming, to increased sediment discharge from a tributary. Due to irrigation since 1900, Muddy Creek has greatly increased its sediment load to East Fork River while adding less than 3% to East Fork River's flood discharge. The response of East Fork River has been to initially adjust only roughness and depth and then after several years to adjust width. Slope has not adjusted yet, but Andrews thinks it will after "a considerable length of time" (1979, p. 92). This is exactly the relative ranking of the rates of change the analysis showed to be necessary for metastable equilibrium.

CONCLUSIONS

The method of analysis presented here for partially specified systems gives qualitative answers to the questions, What is the response of a system when perturbed? and How does that response depend upon the strength of the individual links in the feedback loops? The minimum information necessary is a signed, directed graph showing the system variables and their positive and negative feedback loops. When applied to a simplified closed system of the at-a-station hydraulic geometry of a stream, it predicts that the system is metastable provided the rates of change of friction factor, hydraulic radius, and slope are highest, intermediate, and lowest, respectively. Streams violating this ranking are unstable.

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